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HIGH ENERGY SCATTERING OF NON-CRITICAL STRINGS

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We summarize recent work, in which we consider scattering amplitudes of non-critical strings in the limit where the energy of all external states is large compared to the string tension. We show that the high energy limit is dominated by a saddle point that can be mapped onto an electrostatic equilibrium configuration of an assembly of charges associated with the external states, together with a density of charges arising from the Liouville field. The Liouville charges accumulate on line segments, which produce quadratic branch cuts on the worldsheet. The electrostatics problem is solved for string tree level in terms of hyperelliptic integrals and is given explicitly for the 3- and 4-point functions. For generic values of the central charge, the high energy limit behaves in a string-like fashion, with exponential energy dependence.

1. Introduction

In this lecture, we shall summarize our recent work[1] on the high energy behavior of non-critical bosonic strings. Detailed derivations, as well as extended references to the literature, may be found in [1].

While critical string theory is the prime candidate for a unified theory of matter and gravity, non-critical string theory may be of most interest precisely there where gravity is of no concern. Examples of this situation arise in statistical mechanics systems on random lattices, in Polyakov's mapping of the three dimensional Ising model onto a three dimensional fermionic string, and perhaps even in string type reformulations of Quantum Chromodynamics. All of these physical systems admit some reformulation in terms of non-critical string theory.

Much has been accomplished in the study of non-critical string theory (for reviews on the subject, see e.g. [2]). Our understanding of non-critical string theory to date is particularly advanced for bosonic models with rational central charge $c < 1$. The mapping between discretized random surfaces and random matrices combined with the double scaling limit produces exact results for the correlation functions to all orders in perturbation theory.

Surprisingly, it has turned out to be very difficult to reproduce, with the help of the Liouville approach to non-critical string theory, even the simplest results

obtained via matrix models. For example, the evaluation of scattering amplitudes to tree level presents serious obstacles, which have not, in general, been overcome to date. Despite these obstacles, the Liouville approach to non-critical string theory is of great value. This is especially so since a direct reformulation of fermionic non-critical strings in terms of matrix models is not available, while the generalization from the bosonic to the fermionic string is essentially straightforward within the Liouville approach. Also, the applicability of the matrix model approach seems to be limited to $c \leq 1$ while the Liouville approach again allows for a passage into the $c > 1$ domain. One of the most basic obstacles to reaching beyond the $c = 1$ barrier (within the Liouville field theory approach), is the appearance of conformal primary fields with complex weights, and thus of string states with complex masses. In a series of ingenious papers [3], it was proposed that the string spectrum may be restricted to a subset of "physical states", that have real conformal weights only. This restriction appears to be possible only provided space-time dimension assumes certain special values : 1, 7, 13 and 19 for the bosonic string, and 1, 3, 5 and 7 for the fermionic string.

In our work,[1] we developed calculational methods that allow for an evaluation of scattering amplitudes in non-critical string theory. We proposed an evaluation of non-critical string amplitudes for any complex c, in the limit where the energies of incoming and outgoing string states are all large compared to the (square root of the) string tension. We showed that the Liouville approach lends itself naturally to taking the high energy limit, where the integral representations for the amplitudes become tractable, for any complex value of c. To string tree level, we succeed in producing explicit formulas for the limit in terms of hyper-elliptic integrals. We shall not, at this stage, perform any truncation on the spectrum of states in the non-critical string theory. Thus, our results are applicable to non-critical string theories in general, including those in which the Liouville field is reinterpreted as an extra dimension of space-time.

For string theory in the critical dimension, the high energy limit of scattering amplitudes is dominated by a saddle point in the positions of the vertex operators for external string states, as well as in the moduli of the surface. This problem is equivalent to finding the equilibrium configuration of an array of electrostatic Minkowskian charges (attached to the vertex operators) on a surface of variable shape. In a series of beautiful papers[4], it was shown how the saddle point can be constructed by symmetry arguments, for the four point function, to any order in perturbation theory.

For non-critical string theory, the high energy limit is still dominated by a saddle point, which is equivalent to the equilibrium configuration of an array of (complex) charges on a surface of variable shape. In addition to the charges from the external vertex operators, we now also have charges from the Liouville exponential operator. In fact, the number of Liouville charges on the surface increases linearly with energy and, in the limit of large energy, accumulate onto a continuous charge density. We shall show that this Liouville charge density consists of line segments, producing

quadratic branch cuts on the worldsheet.

We shall solve explicitly the equivalent electrostatics problem for a worldsheet with the topology of a sphere (tree level) in terms of hyper-elliptic functions, and use it to deduce the high energy limit of tree level scattering amplitudes. The solution in this limit is valid for any complex value of the matter central charge c, and we use analytic continuation to define the non-critical string amplitudes throughout the complex c plane. For higher genus topologies, the solution involves quadratic branch cuts of higher genus surfaces, but we shall postpone a full derivation of this case to a later publication.

The main physical result is that, at least for generic values of the matter central charge c, the non-critical amplitudes behave in a string like fashion, with exponential dependence on the energy scale, in the limit of high energy. While it is logically possible that this generic exponential behavior could be absent (and replaced by power-like behavior) at isolated points in the complex c plane, we believe that this is unlikely to occur in the region $1 < c < 25$. It is thus unlikely that the non-critical string theories in this region ever become "quantum field theories".

Much work remains to be done: Our method for obtaining the correlation functions generalizes to loop amplitudes in non–critical string theory. In this case, the problem reduces to a two dimensional electrostatics problem on double coverings of higher genus Riemann surfaces. Also, the continuum approach naturally extends to the supersymmetric case. Perhaps more importantly, the correlation functions may be used to analyze the physics of non-critical string theories. Given the diverse applications of non–critical string theory, this is of great interest. These points are currently under investigation.

2. Liouville Field Theory Approach to Non-Critical Strings

The basic ingredients in the Liouville field theory formulation of bosonic non-critical string theory are as follows. The starting point is a "matter" conformal field theory, with central charge c, and fields $x^\mu(z)$, $\mu = 1.\cdots, c$, describing Poincaré invariant string dynamics in a c-dimensional space-time.

The above conformal field theory is coupled to a quantized worldsheet metric g, which, in conformal gauge, decomposes into the Liouville field $\phi(z)$ and a fiducial metric $\hat{g}(m_j)$ that only depends on the moduli of the Riemann surface Σ, with $g = \hat{g}\exp\{2\phi\}$. The action for the Liouville field is

$$S_L = \frac{1}{4\pi}\int \sqrt{\hat{g}}\left[\frac{1}{2}\phi\Delta_{\hat{g}}\phi - \kappa R_{\hat{g}}\phi + \mu e^{\alpha\phi}\right] \tag{1}$$

Here $R_{\hat{g}}$ is the Gaussian curvature of the metric $\hat{g}$, and the coupling constants κ and α are given in terms of the matter central charge c as follows $3\kappa^2 = 25 - c$. The gravitationally dressed vertex operators may be obtained as follows

$$\mathcal{V}_\delta \equiv \int d^2z\mathcal{P}(\partial x^\mu)e^{ik\cdot x+\beta\phi(z)}, \qquad \beta(\delta) = \frac{-\sqrt{25-c}+\sqrt{1-c+24\delta}}{2\sqrt{3}} \tag{2}$$

Here, $\mathcal{P}$ is a polynomial in the derivatives of x^μ of degree Δ, so that $\delta = \Delta + \frac{1}{2}k^2$. For simplicity, we shall assume that $\mathcal{P}$ is independent of ϕ. The coupling constant α is given by $\alpha = \beta(0)$, and the analytic continuation in c and in the external momentum dictates which branches of the square roots should be chosen.

We shall now compute out the correlation function for N external gravitationally dressed vertex operators. The correlation functions of the matter part are standard, so we concentrate on the evaluation of the Liouville correlation functions. To evaluate the Liouville correlation functions, we follow the procedure of [5]. We split the Liouville field ϕ as follows $\phi = \phi_0 + \varphi$, where ϕ_0 is constant on the worldsheet and φ is orthogonal to constants. The integration splits accordingly, and the integral over ϕ_0 may be carried out as follows

$$\int D_{\hat{g}}\phi\, e^{-S_L} \prod_{j=1}^{N} e^{\beta_j \phi(z_j)} = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} \int D_{\hat{g}}\varphi\; e^{-S'_L} \left(\int \sqrt{\hat{g}}\; e^{\alpha\varphi}\right)^s \prod_{j=1}^{N} e^{\beta_j \varphi(z_j)} \quad (3)$$

The new Liouville action S'_L is obtained by setting $\mu = 0$, and the variable s is a scaling dimension, defined by α, κ, β_j and the genus of the worldsheet h as follows

$$\alpha s = -\kappa(1-h) - \sum_{j=1}^{N} \beta_j \quad (4)$$

In general, s does not have to be integer, not even rational. The prescription of [5] is to proceed and carry out the functional integration over φ as if s were an integer, and then later on continue in s. We are confident that this procedure is reliable in view of the semi-classical analysis carried out in the second reference of [5].

3. The High Energy Limit as a Problem in Electrostatics

To tree level, the worldsheet topology is that of the sphere (or by stereographic projection, of the complex plane), there are no moduli, and all determinant factors are constants. The Green function is the electrostatic potential on the two dimensional plane, given by $G(z,z') = -\ln|z-z'|^2$. Tree-level non-critical amplitudes — evaluated for vertex operators that are exponentials only — then reduce to a simple multiple integral expression

$$\langle \prod_{i=1}^{N} \mathcal{V}_i \rangle = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} \int \prod_{i=1}^{N} d^2 z_i \prod_{\substack{i<j \\ i,j=1}}^{N} |z_i - z_j|^{2u_{ij}} \int \prod_{p=1}^{s} d^2 w_p\, |z_j - w_p|^{-2\alpha\beta_j} \prod_{\substack{p,q=1 \\ p<q}}^{s} |w_p - w_q|^{-2\alpha^2} \quad (5)$$

where $u_{ij} = -\beta_i\beta_j + k_i \cdot k_j$. The problem of evaluating correlation functions in non-critical string theory is seemingly reduced to the problem of computing a finite dimensional multiple integral with respect to $z_i,\ w_a$ over the complex plane. Since s is not, in general, an integer however, the correlation function is not well defined as it stands. From the arguments presented in [5], it is clear that the original expression for the amplitudes is complex analytic in the external momenta and in the central

charge c, even though the intermediate expressions only make sense for integer s. Thus, the results obtained for integer s will have to be analytically continued in s, which can be achieved through a combination of analytic continuation in the external momenta (just as in the critical string) and in the central charge c.

For rational $c < 1$, and to string tree level, it was proposed in [5] to analytically continue in the variable s, using certain rearrangement formulas for ratios of Euler Γ-functions (that are specific to tree level). The validity of this procedure is justified, after the fact, since it produces agreement with results from matrix models. More importantly, agreement can be established from first principles, as was shown in the second reference in [5], using a saddle point approximation in the limit when $\alpha \to 0$, i.e. when $c \to \infty$. We shall take these analyticity properties as a definition for the amplitudes away from $c < 1$ and rational.

For tree level amplitudes, the above multiple integrals are of the same type as those discussed by Selberg and in [6]. The 3-point function was obtained in their work for arbitrary parameters, but results on the 4-point function are limited to $c < 1$ conformal matter. In general, these integrals are not available in explicit form.

We propose to evaluate the non-critical string correlation functions in the limit where the energies and momenta of the external string states all become large compared to the square root of the string tension. We shall define the high energy limit by rescaling all momenta k_i by a common factor $\lambda \to \infty$. We have the asymptotic behavior $k_i \to \lambda k_i,\ \beta_i \to \pm\lambda|k_i| + \mathcal{O}(1)$. The scaling properties of other quantities are easily deduced from the above : u_{ij} scales like λ^2, s scales like λ^1 while c and α scale like λ^0. Notice that external vertex operators always remain conformally invariant under this scaling.

To determine the high energy limit of the non-critical scattering amplitude, we begin by analyzing the high energy behavior of the electrostatic energy function $\mathcal{E}_0$ corresponding to the correlation function of Eq. (5), defined by

$$\mathcal{E}_0(z_i, w_p) = -\sum_{\substack{i,j=1\\ i<j}}^{N} u_{ij} \ln|z_i - z_j|^2 + \sum_{j=1}^{N}\sum_{p=1}^{s} \alpha\beta_j \ln|z_j - w_p|^2 + \sum_{\substack{p,q=1\\ p<q}}^{s} \alpha^2 \ln|w_p - w_q|^2 \tag{6}$$

From the expression for $\mathcal{E}_0$, it can be readily shown that $\mathcal{E}_0$ scales like λ^2 for large λ. This is manifest for the first term in $\mathcal{E}_0$, but the next two terms also scale like λ^2 for large λ. Although the couplings in the second term only scale linearly in λ, the number of Liouville insertion points, s, grows like λ. In the third term, the coupling α scales like $\lambda^0 = 1$, but there are now s^2 Liouville insertion points, so again this term scales like λ^2.

The next ingredient needed in the determination of the high energy limit of the non-critical scattering amplitudes is the degree of dependence of this limit on any specific matter conformal field theory. The most important simplification in this respect comes from the observation that the conformal primary fields $\mathcal{P}_\Delta$ involve and produce only polynomial dependence on the space-time momenta k_i. Thus,

in the high energy limit, where the contributions from the saddle point will be generically exponential (as we shall establish below), we may neglect the polynomial contributions from the vertex functions $\mathcal{P}_\Delta$. Thus, only the exponential vertex operator parts contribute to the high energy limit.

The saddle point equations for the integral are just the equations for electrostatic equilibrium of the associated electrostatics problem. They are given by

$$-\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{2b_{ij}}{z_i-z_j}+\frac{1}{s}\sum_{p=1}^{s}\frac{a_i}{z_i-w_p}=0 \qquad\qquad \frac{1}{s}\sum_{\substack{q\neq p\\ q=1}}^{s}\frac{2}{w_p-w_q}+\sum_{j=1}^{N}\frac{a_j}{w_p-z_j}=0 \quad (7)$$

Here, we have defined parameters $a_i \equiv 2\beta_i/(\alpha s)$ and $b_{ij} \equiv u_{ij}/(\alpha s)^2$ both of which scale like λ^0 in the limit of large λ. Also, each summation over the number of Liouville charges at w_p, $p = 1, \cdots, s$ has been divided by a factor of s, so that the entire equations scale like λ^0 in the limit of large λ.

The variables $\bar{z}_i$ and $\bar{w}_p$ satisfy the same equation (7), with z_i and w_p replaced by $\bar{z}_i$ and $\bar{w}_p$ respectively. When the charges a_i and b_{ij} are real, those respective equations are just the complex conjugates of one another. But when the charges a_i and b_{ij} are taken to be complex, the equations are no longer complex conjugates of one another, and $\bar{z}_i$ and $\bar{w}_p$ at the saddle point are no longer the complex conjugates of z_i and w_p respectively.

We wish to solve the equations (7) in the limit where $s \to \infty$, while keeping a_i, b_{ij} and the number of vertex charges N fixed. The most difficult part of this problem is the solution of the second equation, for the following reasons. Since the number of Liouville charges at w_p tends to ∞, they must accumulate somewhere, possibly at infinity. A priori, the limiting distribution might correspond to two-dimensional regions of charge, to one-dimensional line segments, to isolated points, or even to more exotic arrangements of fractional dimension such as Cantor sets.

We shall start by providing an answer to this question first, by carefully keeping the Liouville charges at w_p isolated, and taking the limit only when completely safe. In [1], we showed that this problem can be solved exactly in the case of the three point function with $N = 3$, and we found there that the Liouville charges accumulate onto a single line segment. More generally, we shall find that the Liouville charges accumulate onto a collection of $N - 2$ curve segments.

4. Solution of the Electrostatics Problem to Tree Level

To study equation (7) for general N, we make use of a complex potential $W(z)$ for the charges z_i and re-express the equation (7) for the Liouville charges at w_p as

$$\frac{1}{s}\sum_{\substack{q=1\\ q\neq p}}^{s}\frac{1}{w_p-w_q}=\frac{1}{2}W'(w_p) \quad \text{where} \quad W(z)\equiv-\sum_{j=1}^{N-1}a_j\ln(z_j-z) \qquad (8)$$

This equation, for general $W(z)$, is just the electrostatics condition for an assembly of s charges in the presence of an external potential $W(z)$ — given here by the

potential generated by the charges at z_i. We also introduce a complex analytic generating function $\omega(z)$, defined by

$$\omega(z) = \frac{1}{s}\sum_{p=1}^{s}\frac{1}{w_p - z} \tag{9}$$

which, physically, is just the electric field produced by the Liouville charges at w_p. Its divergence is obtained by applying the Cauchy-Riemann operator, and yields the (two-dimensional) electric charge density with unit integral over the plane.

One may re-express the set of s equations (8) in terms of the following Riccati equation for $\omega(z)$ by introducing an auxiliary potential $R(z)$

$$\omega^2(z) - \frac{1}{s}\omega'(z) + W'(z)\omega(z) + \frac{1}{4}R(z) = 0, \qquad R(z) \equiv \frac{4}{s}\sum_{p=1}^{s}\frac{W'(w_p) - W'(z)}{w_p - z} \tag{10}$$

For general $W(z)$, it would not be possible to carry out the sum in the definition of $R(z)$ in any simple way. When $W(z)$ is a rational function of z however, as is the case here, $R(z)$ is also rational, with poles at precisely the same locations as $W(z)$:

$$R(z) = \sum_{i=1}^{N-1}\frac{R_i}{z - z_i}, \qquad \text{where} \qquad R_i = \frac{4a_i}{s}\sum_{p=1}^{s}\frac{1}{w_p - z_i} \tag{11}$$

The fact that we have been able to determine the functional form of $R(z)$ explicitly, in terms of a finite number of parameters is perhaps the most important ingredient in our solution of the associated electrostatics problem.

All that precedes is still an exact transcription of the electrostatics equations (7), valid for any finite number of Liouville charges at w_p. We shall now bring about one further simplification by using the approximation in which the number of charges s is large. (Recall that, in the original non-critical string problem, this limit corresponds to high energy of all external string states.)

The potential $W(z)$ is independent of s, while the electric field $\omega(z)$ and the auxiliary potential $R(z)$ converge to finite limits as $s \to \infty$. Thus, the electrostatics equation for $\omega(z)$ of (10) can be simplified in this limit, as the term in $\omega'(z)$ is suppressed by a factor of $1/s$ and may be dropped. Instead of the Riccati equation of (10), we obtain a quadratic equation which is easily solved. We obtain

$$\omega(z) = \frac{1}{2}\left[-W'(z) \pm \sqrt{W'^2(z) - R(z)}\right], \tag{12}$$

The sign in front of the square root should be chosen so that the poles in $\omega(z)$, located at the points z_i, are absent when the charges a_i are all real and positive. Eq. (12) immediately determines the density of Liouville charges; since the solution $\omega(z)$ in (12) is holomorphic away from the possible poles at z_i, and away from the quadratic branch cuts arising from the square root, we see that the Liouville charge accumulate on the branch cuts of ω.

The positions of the associated branch points are most easily exhibited by recasting the solution for $\omega(z)$ as follows :

$$W'(z)^2 - R(z) = Q_{2N-4}(z) \prod_{i=1}^{N-1} (z - z_i)^{-2} \tag{13}$$

Here, $Q_{2N-4}(z)$ is a polynomial in z, which is of degree $2N-4$, in view of the fact that the sum of all R_i vanishes. We find

$$Q_{2N-4}(z) = (a+2)^2 \prod_{k=1}^{2N-4} (z - x_k) \qquad\qquad a = \sum_{i=1}^{N-1} a_i \tag{14}$$

The function $\omega(z)$ thus exhibits $N-2$ branch cuts, $\mathcal{C}_p$, spanned between pairs of branch points x_{2p-1} and x_{2p}, $p = 1, \cdots, N-2$, of $\omega(z)$, which correspond to zeros of the polynomial $Q_{2N-4}(z)$.

Since the configuration of the Liouville charges at w_p is one-dimensional, it is convenient to introduce the linear density $\rho(w)$, defined when w lies on $\mathcal{C}$.

$$\rho(w) = \frac{1}{2\pi} \sqrt{R(w) - W'(w)^2} \tag{15}$$

The requirement that $\mathcal{C}$ lie along branch cuts of ω does not determine the precise position of $\mathcal{C}$. In fact, any analytic curve that joins pairs of branch points would do. From the fact that the Liouville charges at the points w_p are all of unit strength times $1/s$, it follows that the charge density must be real and positive along $\mathcal{C}$. This supplementary condition requires that the position of the branch cut $\mathcal{C}$, supporting the Liouville charges at w_p, must be such that $dw\rho(w)$ is real as w is varied along $\mathcal{C}$. When α and β_i are real, this simply implies that the Liouville charges at w_p are concentrated on the real axis, as was expected. However, the above conditions also provide consistent prescriptions for the case when α, β_i as well as the external momenta, are analytically continued to complex values.

It remains to clarify the physical significance of the constants R_i, which enter the function $R(z)$ in (11) and the polynomial $Q_{2N-4}(z)$ in (13), in the $s \to \infty$ limit. These relations are *automatically* satisfied by the construction of $\omega(z)$, as can be checked easily by taking the limit of (9) when $z \to z_i$. It thus appears that the $N-3$ independent parameters R_i, entering the solution of the second set of equations in (7) are undetermined by these equations.

How can this indeterminacy be understood ? It is easiest to analyze first the case where all z_i, and all a_i are real. By construction, the R_i are then real, as can be seen from (11). When all a_i, $i = 1, \cdots, N-1$ are positive, (and only the compensating charge at ∞ is negative) the possible locations for the Liouville charges are on the $N-2$ line segments $\mathcal{C}_p$, $p = 1, \cdots, N-2$, in between pairs of consecutive positive charges at z_i. However, exactly how the total Liouville charge (which is fixed to be 1) is partitioned among the $N-2$ line segments is not à priori determined. Indeed, the positive Liouville charges cannot cross over from one

line segment into another, since crossing would involve passing through a charge configuration of infinite electrostatic energy when a Liouville charge is on top of a charge z_i. Thus, for any partition of the Liouville charges among the $N-2$ intervals, there must be an equilibrium configuration, and the $N-3$ independent parameters R_i precisely specify the possible partitions of the Liouville charges over the $N-2$ intervals. When the points z_i and the charges a_i are allowed to be complex, the line segments on which the Liouville charges lie can move into the complex plane, but the counting is analogous.

In fact, the values of the parameters R_i are determined by the first set of equations in (7), which give the positions of the external vertex charges z_i, $i = 1, \cdots, N-1$. In the high energy limit, where $s \to \infty$, we have

$$\sum_{\substack{j=1 \\ j\neq i}}^{N-1} \frac{2b_{ij}}{z_i - z_j} + \frac{1}{4}R_i = 0 \tag{16}$$

Therefore, out of the original $N-1$ equations in (16), two correspond to the asymptotic conditions on $\omega(z)$, leaving $N-3$ equations. In view of the analysis of the previous paragraph, only $N-3$ real parameters amongst the $N-3$ complex R_i are determined by the electrostatics equations and the reality conditions, leaving N-3 real parameters undetermined.

5. High Energy Limit from the Electrostatic Energy

The tree level correlation function, in the saddle point approximation, is

$$\langle \prod_{i=1}^{N} \mathcal{V}_i \rangle = \frac{\Gamma(-s)\mu^s}{\alpha(4\pi)^s} e^{-\mathcal{E}_0} \tag{17}$$

where all the quantities are to be evaluated at the saddle point. In particular, the electrostatic equilibrium energy is given in terms of $\rho(w)$ and $W(w)$ by

$$-\frac{\mathcal{E}_0}{2(\alpha s)^2} = \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N-1} b_{ij} \ln|z_i - z_j|^2 + \int_{\mathcal{C}} dw\rho(w)\{W(w)+\bar{W}(w)\} - \frac{1}{2}\int_{\mathcal{C}} dv\rho(v)\int_{\mathcal{C}} dw\rho(w)\ln|v-w|^2 \tag{18}$$

First, the saddle point equations for $\bar{z}_i$ and $\bar{w}_p$ are the same as for the quantities z_i and w_p, even when the charges a_i and b_{ij} are complex. Second, the integration measure $dw\rho(w)$ must be real along the line segments of charge density. We see that, as a result, the entire electrostatic energy is a sum of a contribution from z_i and w_p on the one hand, and the same functional form, evaluated on $\bar{z}_i$ and $\bar{w}_p$ on the other hand. Thus, given the identity of the equations for barred and unbarred quantities, the electrostatic energy is just twice that evaluated on unbarred quantities only. This simplified expression for the electrostatic energy at equilibrium can be recast in terms of the holomorphic potential

$$\Omega(z) = \int_{\mathcal{C}} dw\rho(w)\ln(z-w) \tag{19}$$

of the Liouville charges at w_p. This potential is the analogue of the potential W for the charges z_i, and its derivative is $\omega(z) = \Omega(z)'$. In terms of this function, we may evaluate the electrostatic potential of the w_p-charges in a simplified way:[1]

$$-\frac{\mathcal{E}_0}{2(\alpha s)^2} = \sum_{\substack{i,j=1 \\ i\neq j}}^{N-1} b_{ij}\ln(z_i - z_j) - W_0 - \sum_{i=1}^{N-1} a_i \Omega(z_i) \tag{20}$$

These quantities are now all holomorphic, and as such will not be changed upon continuous changes in the curve $\mathcal{C}$. Thus, any curve $\mathcal{C}$, connecting the branch points can be used in the expression above, which greatly simplifies its calculability.

In fact, it was further shown in [1] that the function Ω itself may be evaluated in terms of Abelian differentials. Let us simply quote the results :

$$\Omega(z) - \Omega(z_0) = -\frac{1}{2}\sum_{i=1}^{N-1} a_i \ln\frac{z - z_i}{z_0 - z_i} - i\pi\int_{z_0}^{z} dw\rho(w) \tag{21}$$

Thus, the electrostatic energy at the saddle point is now computed completely in terms of Abelian integrals on the associated hyperelliptic surface. The three and four point functions may be worked out explicitly, and can be shown to exhibit exponential behavior in u_{ij}[1].

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